Introduction to partial differential equations (NMMA339) – theory

Petr Velička *
lecturer: prof. Mgr. Milan Pokorný, Ph.D., DSc. †

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*petrvel@matfyz.cz
†pokorny@karlin.mff.cuni.cz

1 Question 1

1.1 Part A

Linear 1st-order PDE is the equation

$$\sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i}(x) = f(x)$$

which we consider for $x \in \Omega \subset \mathbb{R}^n$. By its solution we mean a continuously differentiable function $u(x)$ such that the previous equation holds pointwise for all $x \in \Omega$.

Characteristic system for this equation is a system of ODEs $\frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = a_i(x(t))$ for $i = 1, \dots, N$ and $x \in \Omega$.

Theorem 1.1 (characterization of solutions of 1st order linear PDE)**.** *Function $\psi(x_1, \dots, x_N)$ is a solution for a linear homogeneous PDE if and only if it is constant along each of the solutions of the characteristic system.*

Proof. $\Rightarrow$: Let ψ be such that $\sum_{i=1}^{N} a_i(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x_i} = 0$. Let $\{\varphi_1(t), \dots, \varphi_N(t)\}$ be one of its characteristics. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\varphi_1(t), \dots, \varphi_N(t)) = \sum_{i=1}^{N} \frac{\partial\psi}{\partial x_i}(\varphi_1(t), \dots, \varphi_N(t))\frac{\mathrm{d}\varphi_i(t)}{\mathrm{d}t} =$$

$$\sum_{i=1}^{N} \frac{\partial\psi}{\partial x_i}(\varphi_1(t), \dots, \varphi_N(t))a_i(\varphi_1(t), \dots, \varphi_N(t)) = 0.$$

$\Leftarrow$: Let ψ be constant along every characteristic. Choose $\xi \in \Omega$. Since $\{a_i\}_{i=1}^{N}$ are continuous, at least one characteristic goes through the point ξ which corresponds to some interval (t_1, t_2). Fix such characteristic $(\varphi_1(t), \dots, \varphi_N(t))$ and a point τ such that $(\varphi_1(\tau), \dots, \varphi_N(\tau)) = (\xi_1, \dots, \xi_N)$.

Since $\psi(\varphi_1(t), \dots, \varphi_N(t))$ is constant on (t_1, t_2), we have that

$$0 = \frac{\mathrm{d}\psi}{\mathrm{d}t}(\varphi_1(t), \dots, \varphi_N(t)) = \sum_{i=1}^{N} \frac{\partial\psi}{\partial x_i}(\varphi_1(t), \dots, \varphi_N(t))a_i(\varphi_1(t), \dots, \varphi_N(t)).$$

Now it is sufficient to choose $t = \tau$ and we get that ψ is a solution at point ξ. Since this point was chosen arbitrarily, we conclude that it is a solution on the entire Ω. $\square$

1.2 Part B

Fundamental solution of the Poisson equation is the function

$$\mathcal{E}(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, N = 2; \\ \frac{1}{(N-2}\kappa_N|x|^{N-2}, N \geq 3, \end{cases}$$

where κ_N is the measure of the n-dimensional unit ball.

Theorem 1.2 (Three Potentials Theorem)**.** *Let $\omega \subset \mathbb{R}^N$ be a bounded region with at least Lipschitz boundary, $u \in C^2(\bar{\Omega})$ and $\mathcal{E}$ is the fundamental solution of the Poisson equation. Then for all $x \in \Omega$,*

$$u(x) = -\int_\omega \mathcal{E}(x-y)\Delta u(y)dy + \int_{\delta\Omega} (\mathcal{E}(x-y)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \mathcal{E}}{\partial n_y}(x-y))dS(y),$$

where

$$\frac{\partial u}{\partial n}(y) := \nabla u(y)n(y), \frac{\partial \mathcal{E}}{\partial n_y}(x-y) := \nabla_y \mathcal{E}(x-y)n(y)$$

and n is the outer normal to $\delta\Omega$.

Proof. Let $x \in \Omega$ and let $B_\varepsilon(x) \subset \Omega$. Let $V_\varepsilon := \Omega \setminus \overline{B_\varepsilon(x)}$. Using Green's theorem, we obtain

$$\int_{V_\varepsilon} \mathcal{E}(x-y)\Delta u(y)dy = \int_{V_\varepsilon} \delta_y \mathcal{E}(x-y)u(y)dy + \int_{\delta V_\varepsilon} \mathcal{E}(x-y)\frac{\partial u}{\partial n}(y)dS(y) -$$

$$\int_{\delta V_\varepsilon} u(y)\frac{\partial \mathcal{E}}{\partial n_y}(x-y)dS(y).$$

Denote the individual integrals $I_1 = I_2 + I_3 - I_4$ and send ε to zero.

First we can write (the functions in question are "nice" enough)

$$I_1 \to \int_\Omega \mathcal{E}(x-y)\Delta u(y)dy$$

I_2 is obviously zero (fundamental solution).

I_3 we can rewrite in the following manner:

$$I_3 = \int_{\delta\Omega} \mathcal{E}(x-y)\frac{\partial u}{\partial n}(y)dS(y) - \int_{\delta B_\varepsilon(x)} \mathcal{E}(x-y)\frac{\partial u}{\partial n}(y)dS(y) \to$$

$$\int_{\delta\Omega} \mathcal{E}(x-y)\frac{\partial u}{\partial n}(y)dS(y).$$

For the last integral, we can write

$$I_4 = \int_{\delta\Omega} u(y)\frac{\partial \mathcal{E}}{\partial n_y}(x-y)dS(y) - \int_{\delta B_\varepsilon(x)} u(y)\frac{\partial \mathcal{E}}{\partial n_y}(x-y)dS(y) \to$$

$$\int_{\delta\Omega} u(y)\frac{\partial \mathcal{E}}{\partial n_y}(x-y)dS(y) + u(x).$$

From this follows the equality in question. $\square$

2 Question 2

2.1 Part A

Functions $u_1(x_1,\dots,x_n), u_2(x_1,\dots,x_N), u_n(x_1,\dots,x_N)$ are dependent on the set $\bar{O}$ (O is open and bounded), if there exists a continuously differentiable function $F(u_1,\dots,u_n)$ such that

1. F is not identically zero on any region $G \subset \mathbb{R}^n$.
2. for all $x = (x_1,\dots,x_N) \in \bar{O}$ we have that $F(u_1(x),\dots,u_n(x)) = 0$.

Theorem 2.1 (Jacobi's criterion for function dependence)**.** *Let $\Omega \subset \mathbb{R}^N$ be open, $u_i \in C^1(\Omega)$ for $i = 1,\dots,N$. Then $u_1(x),\dots,u_n(x)$ are dependent in Ω if and only if*

$$J_u(x) = \det \begin{pmatrix} \frac{\partial u_1(x)}{\partial x_1} & \cdots & \frac{\partial u_1(x)}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_N(x)}{\partial x_1} & \cdots & \frac{\partial u_N(x)}{\partial x_N} \end{pmatrix} = 0.$$

Theorem 2.2 (Dependence of N solutions)**.** *Let $\psi_1,\dots,\psi_N$ be solutions of a non-trivial linear homogeneous 1st-order PDE in $\Omega \subset \mathbb{R}^N$. Then they are dependent in Ω.*

Proof. Assume that $\psi_1,\dots,\psi_N$ are not dependent in Ω. Then there exists a point $x_0 \in \Omega$ such that $J_\psi(x_0) \neq 0$. From continuity of solutions we have that the determinant in question is non-zero on some neighborhood $U(x_0)$. This implies that the system $\begin{pmatrix} \frac{\partial u_1(x)}{\partial x_1} & \cdots & \frac{\partial u_1(x)}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_N(x)}{\partial x_1} & \cdots & \frac{\partial u_N(x)}{\partial x_N} \end{pmatrix} y = 0$ only has a trivial solution, which is in contradiction with the assumption that our original PDE was non-trivial. □

2.2 Part B

Consider the following problem: $\frac{\partial^2 v(t,x;s)}{\partial t^2} - c^2 \frac{\partial^2 v(t,x;s)}{\partial x^2} = 0$, $v(s,x;s) = 0$ and $\left(\frac{\partial f(t,x;s)}{\partial t}\right)_{t=s} = f(t,x)$.

Then $u(t,x) := \int_0^t v(t,x;s)ds$ solves general wave equation with RHS f.

For $N = 1$ we have $v(t,x;s) = \frac{1}{2c}\int_{B_{c(t-s)}(x)} f(s,y)dy$ and thus

$$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)dyds = \frac{1}{2c}\int_0^t \int_{x-c\tau}^{x+c\tau} f(t-\tau,y)dyd\tau.$$

For $N = 2$ we have $v(t,x;s) = \frac{1}{2\pi c}\int_{B_{c(t-s)}(x)} \frac{f(s,y)}{\sqrt{c^2(t-s)^2-|y-x|^2}}dy$. Thus

$$u(t,x) = \frac{1}{2\pi c}\int_{B_{c(t-s)}(x)} \frac{f(s,y)}{\sqrt{c^2(t-s)^2-|y-x|^2}}dyds.$$

3 Question 3

3.1 Part A

Linear 1st-order PDE is the equation

$$\sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i}(x) = f(x)$$

which we consider for $x \in \Omega \subset \mathbb{R}^n$. By its solution we mean a continuously differentiable function $u(x)$ such that the previous equation holds pointwise for all $x \in \Omega$.

Functions $u_1(x_1, \dots, x_n), u_2(x_1, \dots, x_N), u_n(x_1, \dots, x_N)$ are dependent on the set $\bar{O}$ (O is open and bounded), if there exists a continuously differentiable function $F(u_1, \dots, u_n)$ such that

1. F is not identically zero on any region $G \subset \mathbb{R}^n$.
2. for all $x = (x_1, \dots, x_N) \in \bar{O}$ we have that $F(u_1(x), \dots, u_n(x)) = 0$.

Let $\psi_1, \dots, \psi_{N-1}$ are solutions of the 1st-order PDE. For every subregion $\Omega' \subset \Omega$ let there be a point $x_0 \in \Omega'$ such that the matrix

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_{N-1}}{\partial x_1} & \cdots & \frac{\partial \psi_{N-1}}{\partial x_N} \end{pmatrix}$$

has rank $N-1$. Then ψ_1, ψ_{N-1} make the fundamental system of the linear PDE.

Theorem 3.1 (Maximum Number of Independent Solutions)**.** *Let there exist a point $x \in \Omega'$ for every subregion $\Omega' \subset \Omega$ such that $\sum |a_i(x)| > 0$. Let $\psi_1, \dots, \psi_{N-1}$ be the fundamental system of said PDE. Then $\theta \in C^1(\Omega)$ solves the PDE iff $\psi_1, \dots, \psi_{N-1}, \theta$ are dependent in Ω.*

Proof. $\implies$ was already proven before $\impliedby$: We assume that $J_{\vec{\psi},\theta} = 0$. Therefore the system $y_1 \frac{\partial \psi_1}{\partial x_i} + \cdots + y_{N-1} \frac{\partial \psi_{N-1}}{\partial x_i} + y_N \frac{\partial \theta}{\partial x_i} = 0$ for $i = 1, \dots, N$ has a non-trivial solution for every $x \in \Omega$. Multiplying this equation by $a_i(x)$ and summing all of the together gives us

$$\sum_j y_j(x) \sum_i a_i(x) \frac{\partial \psi_j}{\partial x_i} + y_N(x) \sum_i a_i(x) \frac{\partial \theta}{\partial x_i} = 0.$$

Since ψ_i solve the equation, we get that

$$y_N(x) \sum_i a_i(x) \frac{\partial \theta}{\partial x_i} = 0.$$

Assume that θ does not solve are equation. Therefore we have an x_0 such that $\sum_{i=1}^{N} a_i(x_0)\frac{\partial\theta(x_0)}{\partial x_i} \neq 0$. By the usual notion of continuity we get that there exists a neighborhood of x_0 where this formula is non-zero. Therefore $y_N = 0$ on some $U(x_0)$. Therefore $y_1\frac{\partial\psi_1}{\partial x_i} + \cdots + y_{N-1}\frac{\partial\psi_{N-1}}{\partial x_i} = 0$. Since all the original functions are independent, there is a point x_1 where the system only has a trivial solution and we get that $J_{\vec{\phi},\theta}(x_1) \neq 0$, which contradicts our assumption of independence. $\square$

3.2 Part B

Consider the following system: $-\Delta u = f$ on some Ω, $u = g$ on $\delta\Omega$. If Ω is bounded, we talk about the inner Dirichlet problem, if $\mathbb{R}^n \setminus \bar{\Omega}$ is bounded, we talk about the outer Dirichlet problem (in this case we only care about functions $u(x) = O\left(\frac{1}{|x|^{N-2}}\right)$ for $|x| \to \infty$.

For the uniqueness of the solution for the inner problem, we only need to prove that the zero problem for $f = g = 0$ only has the trivial solution $u = 0$, which in fact follows from the weak maximum principle ($0 = \min_{\delta\Omega} u = \min_{\Omega} u \leq \max_{\Omega} u = \max_{\delta\Omega} u = 0$).

Now we shall prove the uniqueness for the outer problem. Assume that u_1 and u_2 are two solutions. Then $u_1 - u_2 =: w$ solves $\Delta w = 0$ on Ω, $w = 0$ on the boundary and, $w(x) = O\left(\frac{1}{|x|^{N-2}}\right)$.

First assume that $N \geq 3$, therefore $w(x) \to 0$ as $|x|$ goes to infinity. Let $x_0 \in \Omega$ and $\varepsilon > 0$ be given. Then there exists $R > 0$ such that $x_0, G \subset B_R$, where $G := \mathbb{R}^N \setminus \bar{\Omega}$ and $|w| \leq \varepsilon$ on δB_R. Now apply the maximum principle for the set $\Omega_R := \Omega \cup B_R$. We have that $w \leq \varepsilon$ on Ω_R. The same can be said for $-w$, which means that we are done.

Let now $N = 2$. WLOG let the origin be in $G := \mathbb{R}^N \setminus \bar{\Omega}$. Fix an open ball $B_R \subset G$. Define a mapping $F : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$ defined as $F(x) = x' := \frac{xR^2}{|x|^2}$. This means that x and $F(x)$ lie on the same half-line originating in the origin. Then F maps Ω to some region $\Omega^* \subset B_R$, in fact, $F(\Omega) = \Omega^* \setminus \{0\}$. Define $w'(x') = w(F^{-1}(x')) = w(x)$. Obviously $w' = 0$ on $\delta\Omega^*$. We will show that w' is harmonic on $\Omega^* \setminus \{0\}$. Set $r = |x|$ ad $\rho = |x'|$. Then set $\tilde{w}(r, \phi) = w(x_1, x_2)$ and $\tilde{w}'(\rho, \phi') = w'(x_1', x_2')$ and moving to the polar coordinates we get the desired equality. Now the function is obviously bounded, therefore we can extend it harmonically. By what was already proven for the inner problem, this means that $w' = 0$, which in turn implies that $w = 0$ on $\bar{\Omega}$.

4 Question 4

4.1 Part A

Functions $u_1(x_1, \dots, x_n), u_2(x_1, \dots, x_N), u_n(x_1, \dots, x_N)$ are dependent on the set $\bar{O}$ (O is open and bounded), if there exists a continuously differentiable function $F(u_1, \dots, u_n)$ such that

1. F is not identically zero on any region $G \subset \mathbb{R}^n$.
2. for all $x = (x_1, \dots, x_N) \in \bar{O}$ we have that $F(u_1(x), \dots, u_n(x)) = 0$.

Let $\psi_1, \dots, \psi_{N-1}$ are solutions of the 1st-order PDE. For every subregion $\Omega' \subset \Omega$ let there be a point $x_0 \in \Omega'$ such that the matrix

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_1} & \dots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_{N-1}}{\partial x_1} & \dots & \frac{\partial \psi_{N-1}}{\partial x_N} \end{pmatrix}$$

has rank $N-1$. Then ψ_1, ψ_{N-1} make the fundamental system of the linear PDE.

Theorem 4.1. *Let $m < N-1$, ψ_i are C^1 for $i = 1, \dots, m$ and solve the ODE such that*

$$\frac{D(\psi_1, \dots, \psi_m)}{D(x_1, \dots, x_m)}(x_1, \dots, x_m) \neq 0.$$

Let $\tilde{x}_i = \psi_i(x), i \leq m$ and $\tilde{x}_i = x_i$ otherwise. Let $\tilde{\Omega} = \tilde{x}(\Omega)$ and $\tilde{a}_i(\tilde{x}_1, \dots, \tilde{x}_N) = a_i(x_1(\tilde{x}), \dots, x_n(\tilde{x}))$.

Now let $\tilde{\psi}_{m+1}, \dots, \tilde{\psi}_{N-1}$ be solutions of the equation

$$\sum_{i=m+1}^{N} \tilde{a}_i(\tilde{x}) \frac{\partial \tilde{\psi}}{\partial \tilde{x}_i} = 0.$$

Then functions $\psi_1, \dots, \psi_{N-1}$ ($\psi_k(x) = \tilde{\psi}_k(\tilde{x}(x))$) make up the fundamental system of the PDE.

Proof. First let's show that the functions ψ_k solve the equation.

$$\frac{\partial \psi_k}{\partial x_i} = \sum_j^m \frac{\partial \tilde{\psi}_k}{\partial \tilde{x}_j} \frac{\partial \tilde{x}_j}{\partial x_i} + \sum_j^N \frac{\partial \tilde{\psi}_k}{\partial \tilde{x}_j} \frac{\partial \tilde{x}_j}{\partial x_i}$$

Multiplying this equation by $a_i(x)$ and summing them up across all is, we get

$$\sum_i a_i(x) \frac{\partial \psi_k}{\partial x_i} = \sum_j^m \frac{\partial \tilde{\psi}_k(\tilde{x}(x))}{\partial \tilde{x}_j} \sum_i \frac{\partial \psi_j(x)}{\partial x_i} a_i(x) + \sum_j^N \tilde{a}_j(\tilde{x}) \frac{\partial \tilde{\psi}}{\partial \tilde{x}_j} = 0.$$

It remains to prove that the set is indeed the fundamental system. This however comes from rewriting the determinant in question using the chain rule as two determinants of the non-trivial solutions given in the statement. $\square$

4.2 Part B

We are given the equation $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f$ (on $(0,T] \times \Omega$), $u(0,x) = g(x)$, $\frac{\partial u(0,x)}{\partial t} = h(x)$ and

- Dirichlet: $u(t,x) = w(t,x)$ on the boundary $(0,T] \times \delta\Omega$);
- Neumann: $\frac{\partial u}{\partial \vec{n}} = (\nabla u \cdot \vec{n})(t,x) = w(t,x)$ on the same boundary.

For Ω "behaved" enough ($C^{0,1}$ or C^1 boundary for D. and N. respectively), the solution $u \in C^2$ is determined uniquely (if exists).

Proof. Let u_1 and u_2 be two such solutions. Then $u := u_1 = u_2$ solves the following problem:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0$$

and $u = 0$ ($\frac{\partial u}{\partial \vec{n}} = 0$) on $(0,T] \times \delta\Omega$.

Now fix $T_0 \in (0,T)$. For every $t \in (0,T_0)$ multiply the PDE by $\frac{\partial u}{\partial t}$ and integrate:

$$0 = \int_\Omega (\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u) \frac{\partial u}{\partial t} dx = \int_\Omega \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} dx - c^2 \int_\Omega \Delta u \frac{\partial u}{\partial t} dx.$$

Using differentiation under the integral sign and Green's identities, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} c^2 \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |\nabla u|^2 dx = 0.$$

Integrating over $(0,T_0)$, we get

$$\frac{1}{2} \int_\Omega \left(\frac{\partial u(T_0,x)}{\partial t} \right)^2 dx + \frac{1}{2} c^2 \int_\Omega |\nabla u(T_0,x)|^2 dx = 0.$$

From this (both integrals must be zero) we get that $\frac{\partial u}{\partial t} = 0$ and also $u = 0$ on the entire region Ω. $\square$